Series Solution near an ordinary Point :-

Suppose $x=0$ is an ordinary point of the linear equation:

$$b_0(x)\,y'' + b_1(x)\,y' + b_2(x)\,y = 0 \qquad \text{——} (1)$$

that have a solution $y = \sum_{n=0}^{\infty} a_n x^n$ —— (2)

that contains two arbitrary constants namely a_0 and a_1 and converges inside a circle with center at $x=0$ and extending out to the singular point nearest $x=0$. If the differential equation has no singular points in the finite plane, then the solution (2) is valid for all finite x. There remains of course, the job of finding an for $n \geq 2$. That is a major part of our work in solving a particular equation.

In solving numerical equations, the technique employed in the following examples will be found useful.

Ex (4)∞ If the eq. $y'' - y = 0$ then show that $a_{n+2} = \frac{a_n}{(n+1)(n+2)}$ for $n = 0, 1, 2, \ldots$

Solution∞

let $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

the eq. $y'' - y = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n+2=2}^{\infty} (n+2)(n-1+2) a_{n+2} x^{(n+2)-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} - a_n] x^n = 0$$

Hence,

$$(n+1)(n+2) a_{n+2} = a_n \Rightarrow a_{n+2} = \frac{a_n}{(n+1)(n+2)}$$

Ex(5)80 use the Power Series to Solve the differential eq.

① $x^2y'' + xy' + x^2y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad , y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad , \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

replace n by n-2

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + a_1 x + \sum_{n=2}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n(n-1) a_n + n a_n + a_{n-2} \right) x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n^2 a_n - n a_n + n a_n + a_{n-2} \right) x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n^2 a_n + a_{n-2} \right) x^n = 0$$

Then $a_1 = 0$ For $n \geq 0$

$$n^2 a_n + a_{n-2} = 0$$

$$a_{n-2} = -n^2 a_n$$

$$a_n = \frac{-1}{n^2} a_{n-2}$$

Because $a_1 = 0$ then all odd term are zero.

Then for even value we have

$$a_2 = \frac{-1}{2^2} a_0$$

$$a_4 = \frac{-1}{4^2} a_2 = \frac{-1}{4^2} \cdot \frac{-1}{2^2} a_0$$

$$a_6 = \frac{-1}{6^2} a_4 = \frac{-1}{6^2} \cdot \frac{-1}{4^2} \cdot \frac{-1}{2^2} a_0$$

$$\vdots$$

In general

$$a_{2k} = \frac{(-1)^k}{(2)^{2k} \cdot (k!)^2} a_0$$

$$\therefore y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2)^{2k} (k!)^2} x^{2k} .$$

② $y'' - x^2 y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad , \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad , \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{replace } n \text{ by } n+2} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+2}}_{\text{replace } n \text{ by } n-2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left((n+2)(n+1) a_{n+2} - a_{n-2} \right) x^n = 0$$

$$2a_2 = 0$$

$$6a_3 = 0$$

$$(n+2)(n+1) a_{n+2} - a_{n-2} = 0$$

$$(n+2)(n+1) a_{n+2} = a_{n-2}$$

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)}$$

we Conclude

$$a_2 = 0$$

$$a_3 = 0$$

$$a_4 = \frac{a_0}{12}$$

$$a_5 = \frac{a_1}{20}$$

$$y(x) = a_0 + a_1 x + \frac{a_0}{12} x^4 + \frac{a_1}{20} x^5 + \cdots$$

③ $y'' - xy' + 4y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \,, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \,, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

replace n by n+2

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + 4a_0 + \sum_{n=1}^{\infty} 4 a_n x^n = 0$$

$$2a_2 + 4a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 4a_n x^n = 0$$

$$2a_2 + 4a_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} - n a_n + 4a_n\right)x^n = 0$$

$$2a_2 + 4a_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} - (n-4)a_n\right)x^n = 0$$

$$2a_2 + 4a_0 = 0$$

$$(n+2)(n+1)a_{n+2} - (n-4)a_n = 0$$

$$a_2 = -2a_0$$

$$a_{n+2} = \frac{(n-4)}{(n+2)(n+1)} a_n$$

$$\vdots$$

④ $y'' + 3xy' + 3y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} 3 a_n x^n = 0$$

replace n by n+2

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} 3 a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 3n a_n x^n + 3a_0 + \sum_{n=1}^{\infty} 3 a_n x^n = 0$$

$$2a_2 + 3a_0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=1}^{\infty} 3 a_n x^n = 0$$

$$2a_2 + 3a_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} + (3n+3) a_n \right) x^n = 0$$

$2a_2 + 3a_0 = 0$

$2a_2 = -3a_0$

$a_2 = \frac{-3}{2} a_0$

$$(n+2)(n+1)a_{n+2} + (3n+3)a_n = 0$$

$$a_{n+2} = \frac{-(3n+3)}{(n+2)(n+1)} a_n$$

⋮

Exercises To Solve [2]80

① $y'' + 2xy' + 5y = 0$

② $(1+4x^2)y'' - 8y = 0$

③ Solve the equation $y'' + 4y = 0$ near the ordinary point $x = 0$.

④ Find two linearly independent solution, valid for $x > 0$, $xy'' + y = 0$

⑤ $y'' - 2xy' + y = 0$

⑥ Use power series to solve the differential eq.

$y'' - 6y = 0$

4. Applied

Bessels Equation with index an integer

In Bessel's equation: $x^2 y'' + xy' + (x^2 - \ell^2) y = 0$

let $y = \sum_{n=0}^{\infty} a_n x^{n+m}$

$$y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

$$x^2 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2} + x \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ (x^2 - \ell^2) \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+m+2}}_{n = n-2} - \ell^2 \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m} - \ell^2 \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+m)(n+m-1) + (n+m) \right] a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m} - \ell^2 \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+m)\left[n+m-1+1 \right] \right] a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m} - \ell^2 \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} \left((n+m)^2 - \ell^2 \right) a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m} = 0$$

$$(m^2-\ell^2)a_0 x^m + ((1+m)^2-\ell^2)a_1 x^{m+1} + \sum_{n=2}^{\infty}((n+m)^2-\ell^2)a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m} = 0$$

$$(m^2-\ell^2)a_0 = 0 \Rightarrow m^2-\ell^2 = 0 \Rightarrow m^2 = \ell^2 \Rightarrow m = \pm\ell$$

$$((1+m)^2-\ell^2)a_1 = 0 \Rightarrow (1+2m+m^2-\ell^2)a_n = 0$$

نعوض $m=\ell$ فقط موجب

$$(1+2\ell+\ell^2-\ell^2)a_1 \Rightarrow (1+2\ell)a_1 = 0 \longrightarrow a_1 = 0$$

$$\sum_{n=2}^{\infty}\left(a_n[(n+m)^2-\ell^2] + a_{n-2}\right)x^{n+m} = 0$$

$$(n^2+2nm+m^2-\ell^2)a_n + a_{n-2} = 0$$

$$(n^2+2n\ell)a_n + a_{n-2} = 0$$

$$(n^2+2n\ell)a_n = -a_{n-2}$$

$$a_n = \frac{-a_{n-2}}{(n^2+2n\ell)}$$

2.

Applied

for $n = 2$

$$a_2 = \frac{-a_0}{(4+4\ell)} = \frac{-a_0}{2(2+2\ell)} \qquad \frac{-a_{n-2}}{n(n+2\ell)} = \frac{-a_0}{2^2(1+\ell)}$$

for $n = 4$

$$a_4 = \frac{-a_2}{4(4+2\ell)} = \frac{-a_2}{2^2 2(2+\ell)} = \frac{a_0}{a^4 \cdot 2!(2+\ell)(1+\ell)}$$

for $n = 6$

$$a_6 = \frac{-a_4}{6 \cdot 2(3+\ell)} = \frac{-a_0}{2^6 \cdot 3!(3+\ell)(2+\ell)(1+\ell)}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$= x^{\ell} \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{\ell}\left[a_0 + a_2 x^2 + a_4 x^4 + \cdots\right]$$

$$= a_0 x^{\ell}\left[1 - \frac{x^2}{2^2(1+\ell)} + \frac{x^4}{2^4 \cdot 2!(2+\ell)(1+\ell)} - \frac{x^6}{2^6 \cdot 3!(3+\ell)(2+\ell)(1+\ell)} + \cdots\right]\frac{\ell!}{\ell!}$$

$$= a_0 \ell! x^{\ell}\left(\frac{1}{\ell!} - \frac{\left(\frac{x}{2}\right)^2}{(1+\ell)\ell!} + \frac{\left(\frac{x}{2}\right)^4}{2!(2+\ell)(1+\ell)\ell!} - \frac{\left(\frac{x}{2}\right)^6}{3!(3+\ell)(2+\ell)(1+\ell)\ell!} + \cdots\right.$$

$$\frac{2^\ell}{2^\ell} x^\ell = 2^\ell \binom{\frac{x}{2}}{\ell}$$

$$= a_0 \ell! 2^\ell \left(\frac{x}{2}\right)^\ell \left(\frac{1}{\ell!} - \cdots - \right)$$

$$= a_0 \ell! 2^\ell \left(\frac{\left(\frac{x}{2}\right)^\ell}{0!\,\ell!} - \frac{\left(\frac{x}{2}\right)^{\ell+2}}{1!\,(1+\ell)!} + \underbrace{\frac{\left(\frac{x}{2}\right)^{4+\ell}}{2!\,(2+\ell)!} - \frac{\left(\frac{x}{2}\right)^{6+\ell}}{3!\,(3+\ell)!}} \right.$$

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$$\left. + \cdots + (-1)^n \frac{\left(\frac{x}{2}\right)^{2n+\ell}}{n!\,(n+\ell)!} \right)$$

$$= a_0 \ell! 2^\ell \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n+\ell}}{n!\,(n+\ell)!}}_{\text{Bessel's function}}$$

Applied

1) Fourier series

has the form

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx \quad \text{--- (1)}$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin nx$$

$$\frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (2)}$$

2) properties of fourier series

1) $\int_0^{2\pi} \sin nx \, dx = 0$

2) $\int_0^{2\pi} \cos nx \, dx = 0$

3) $\int_0^{2\pi} \sin^2 nx \, dx = \pi$

4) $\int_0^{2\pi} \cos^2 nx \, dx = \pi$

5) $\int_0^{2\pi} \sin nx \sin mx \, dx = 0$

6) $\int_0^{2\pi} \cos nx \cos mx \, dx = 0$

7) $\int_0^{2\pi} \sin nx \cos mx \, dx = 0$

8) $\int_0^{2\pi} \sin nx \cos nx \, dx = 0$

9) $\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where $v_1 = \int v \, dx$, $v_2 = \int v_1 \, dx$ and so on

$u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$

3) Determination of Fourier coefficients

Euler's Formulae

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + a_n\cos nx + b_1\sin x + \cdots + b_n\sin nx \quad \cdots (1)$$

i) To find a_0 : integrate both sides of eq (1) from $x = 0$ To $x = 2\pi$

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi} a_1\cos x\,dx + \cdots + \int_0^{2\pi} a_n\cos nx\,dx$$

$$+ \int_0^{2\pi} b_1\sin x\,dx + \int_0^{2\pi} b_2\sin 2x + \cdots + \int_0^{2\pi} a_n\sin nx\,dx$$

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx \quad , (\text{other integrals} = 0)$$

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\left[x\right|_0^{2\pi} = \frac{a_0}{2}\left[\pi - 0\right]$$

$$\int_0^{2\pi} f(x)\,dx = a_0\,\pi$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

2/ Applied 1112F

ii) To find a_n : Multiply each side of (3) by $\cos nx$ and integrate from $x=0$ to $x=2\pi$

$$\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos nx\,dx + a_1\int_0^{2\pi}\cos x\cos nx + a_2\int_0^{2\pi}\cos 2x\cos nx$$

$$+\cdots + a_n\int_0^{2\pi}\cos^2 nx\,dx + b_1\int_0^{2\pi}\sin x\cos nx + b_2\int_0^{2\pi}\sin 2x\cos nx$$

$$+\cdots + b_n\int_0^{2\pi}\sin nx\cos nx\,dx + \cdots$$

$$\int_0^{2\pi} f(x)\cos nx\,dx = a_n\int_0^{2\pi}\cos^2 nx\,dx$$

$$= a_n\int_0^{2\pi}\frac{1+\cos(2nx)}{2}\,dx$$

$$= a_n\int_0^{2\pi}\frac{1}{2}\,dx + \frac{1}{2}\int_0^{2\pi}\cos 2nx\,dx$$

$$= a_n\left[\frac{x}{2} + \frac{\sin 2nx}{4}\right]_0^{2\pi}$$

$$\int_0^{2\pi} f(x)\cos nx\,dx = a_n\left[\frac{2\pi}{2}\right] = a_n\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

By taking $n=1, 2, 3, \ldots$ we can find values of $a_1, a_2, \ldots$

iii) To find b_n multiply each side of (1) by $\sin nx$ and integrate from $x=0$ to $x=2\pi$

$$\int_0^{2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\sin nx\,dx + a_1\int_0^{2\pi}\sin nx\cos x\,dx + a_2\int_0^{2\pi}\sin nx\cos 2x\,dx$$

$$+\cdots + a_n\int_0^{2\pi}\sin nx\cos nx\,dx + \cdots + b_1\int_0^{2\pi}\sin x\sin nx\,dx$$

$$+ b_2\int_0^{2\pi}\sin 2x\sin nx\,dx + \cdots + b_n\int_0^{2\pi}\sin^2 nx\,dx + \cdots$$

$$\int_0^{2\pi} f(x)\sin nx\,dx = b_n\int_0^{2\pi}\sin^2 nx\,dx$$

$$= b_n\int_0^{2\pi}\frac{1-\cos 2nx}{2}\,dx$$

$$= \pi b_n$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

11/12/F

3/ Applied

Example: find the Fourier series representing

$f(x) = x, \quad 0 \le x < 2\pi$

and sketch its graph from $x = -4\pi$ to $x = 4\pi$

Sol:- let

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots + b_1 \sin x + \cdots + b_n \sin nx + \cdots$$

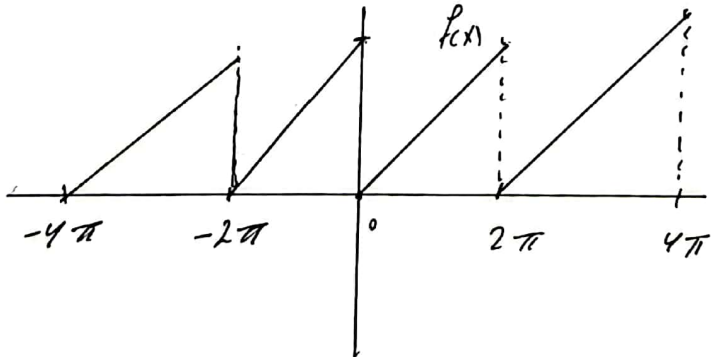
$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} x\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cos nx\, dx = \frac{1}{\pi} \left(x \cdot \frac{\sin nx}{n} + \frac{1}{n^2} \cos nx \right) \Big|_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx = \frac{1}{\pi} \int_0^{2\pi} x \sin nx\, dx = \frac{1}{\pi} \left(-x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right)_0^{2\pi}$$

$$b_n = \frac{1}{\pi} \left(\frac{-2\pi \cdot \cos 2n\pi}{n} \right) = \frac{1}{\pi} \left(\frac{-2\pi}{n} \right) = = \frac{-2}{n}$$

$$f(x) = \pi - 2 \left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \right]$$



Applied 11/10

Ex: Give that $f(x) = x + x^2$ for $-\pi < x < \pi$

find the Fourier expression of $f(x)$ Reduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Sol

$$x + x^2 = \frac{a_0}{2} + a_1 \cos x + \cdots a_n \cos nx + b_1 \sin + \cdots + b_n \sin n + \cdots$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\, 2 \int_0^{\pi} (x + x^2)\, dx = \frac{2}{\pi} \int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi} \left[\pi^2 \frac{\sin n\pi}{n} - 2\pi \left(-\frac{\cos n\pi}{n^2} \right) + 2 \left(-\frac{\sin n\pi}{n^3} \right) \right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx\, dx = \frac{2}{\pi} \left[-(\pi) \frac{\cos n\pi}{n} \right) + \frac{2 \sin n\pi}{n^2}$$

$$= \frac{2}{n} (-1)^{n+1}$$

Substituting the values of a_0, a_n, b_n in eq (1) we get

$$x + x^2 = \frac{1}{2}\frac{2\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \dots\right]$$
$$+ 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right]$$

Put $x = \pi$

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\cos\pi + \frac{1}{4}\cos 2\pi + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right] \quad *$$

Put $\pi = -\pi$

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right] \quad **$$

adding $*$ $**$

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right]$$

$$2\pi^2 - \frac{2\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

1/

Applied

ex: find the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

Sol:- $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -dx + \int_{-\pi/2}^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} dx\right] = 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nx\,dx + \int_{\pi/2}^{\pi} \cos nx\,dx\right] = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\sin nx\,dx + \int_{\pi/2}^{\pi} \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{-\pi/2}\right] + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$\frac{1}{\pi}\left[\frac{\cos n\frac{-\pi}{2}}{n} - \frac{\cos n(-\pi)}{2}\right] + \frac{1}{\pi}\left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{\pi}{2}n}{2}\right]$$

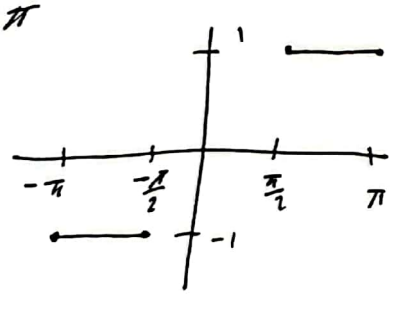
$$= \frac{1}{n\pi}\left[2\cos\frac{n\pi}{2} - 2\cos n\pi\right]$$

$$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$b_1 = \frac{2}{\pi}$

$b_2 = -\frac{2}{2\pi}$

$b_3 = \frac{2}{3\pi}$



putting the values a_0, a_n, b_n in eq (1) we get

$$f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \cdots\right]$$

Find the Fourier series for the periodic function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x)$$

Sol:-

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx \int_{0}^{\pi} x\,dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{0}^{\pi} x\cos nx\,dx\right] = \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{-2}{n^2\pi} \quad \text{when } n \text{ is odd}$$

$$= 0 \quad \text{when } n \text{ is even}$$

2/

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\Big|_0^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n} \right] = \frac{(-1)^{n+1}}{n}$$

Substituting the values of a_0, b_n, a_n in eq (1) we get

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right]$$

$$+ \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \right]$$

2) Discontinuous function

At a point of discontinuity Fourier Series gives the value of $f(x)$ as the arithmetic mean of left and right limits

At the point of discontinuity, $x = c$

At $x = c$ $f(x) = \frac{1}{2}\left[f(c-0) + f(c+0) \right]$

ex: Find the Fourier series for $f(x)$ if

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Sol:-

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_{0}^{\pi} x\,dx \right] = \frac{1}{\pi}\left[-\pi(x)\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_{0}^{\pi} \right]$$

$$a_0 = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{0} + \left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right)\Big|_{0}^{\pi} \right] = \frac{1}{\pi n^2}\left(\cos n\pi - 1\right)$$

$$a_n = \frac{1}{n^2\pi}\left[(-1) + (-1) \right] = \frac{-2}{n^2\pi}$$ where is odd

3/

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{1}{n}\left[1 - 2\cos n\pi\right] = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$$= \begin{cases} \frac{3}{n} & \text{when } n \text{ is odd} \\ -\frac{1}{n} & \text{when } n \text{ is even} \end{cases}$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

$$+ 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \quad (2)$$

putting $x = 0$ in (2) we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \quad (3)$$

now $f(x)$ is discontinuous at $x = 0$

But $f(0-0) = -\pi$ and $f(0+0) = 0$

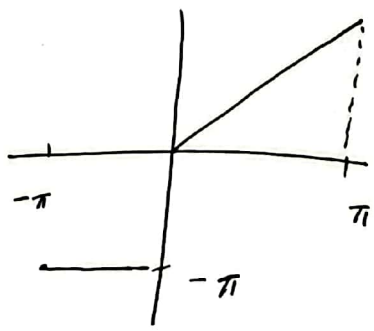
$-\pi$ 1 معادلة

$x=0$ 2 معادلة

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = -\frac{\pi}{2}$$

from eq (3)

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$



$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Ex: find the Fourier Series expansion of the periodic function of period 2π.

$$f(x) = \begin{cases} x & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - 1 & \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

Sol:- $f(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + b_1 \sin x + \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - 1)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi-1)x\right]_{\pi/2}^{3\pi/2} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - 1)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\,\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\,\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\pi}{2}\,\frac{\sin n\pi}{n} - \frac{\cos\frac{n\pi}{2}}{n^2}\right] +$$

$$\frac{1}{\pi}\left[-\frac{\pi}{2}\,\frac{\sin\frac{3n\pi}{2}}{n} - \frac{\cos\frac{3n\pi}{2}}{n^2} - \frac{\pi}{2}\,\frac{\sin n\pi/n}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2}+\sin\frac{n\pi}{2}\right)-\frac{1}{n^2}\left(\cos\frac{3n\pi}{2}-\cos n\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2}+\frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right] = 0$$

$$\sin\theta+\sin\beta = 2\sin\frac{\theta+\beta}{2}\cos\frac{\theta-\beta}{2}$$

$$\cos\theta-\cos\beta = -2\sin\frac{\theta+\beta}{2}\sin\frac{\theta-\beta}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi-1)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)-\frac{\sin nx}{n^2}\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]+\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{3\sin\frac{n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

2/.

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2, \ldots b_1, b_2$ we get

$$f(x) = \frac{4}{\pi}\left[\frac{\sin nx}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right]$$

ex: find the fourier series of the function defined as

$$f(x) = \begin{cases} x + \pi & \text{for } 0 < x < \pi \\ -x - \pi & \text{for } -\pi < x < 0 \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$

Sol:- $f(x) = \frac{a_0}{2} + a_1 \cos + \cdots + b_1 \sin x + \cdots$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} (-x-\pi)\,dx + \frac{1}{\pi} \int_{0}^{\pi} (x+\pi)\,dx$$

$$= \frac{1}{\pi} \left(-\frac{x^2}{2} - \pi x \right)_{-\pi}^{0} + \frac{1}{\pi} \left(\frac{x^2}{2} + \pi x \right)_{0}^{\pi}$$

$$= \frac{1}{\pi} \left(\frac{\pi^2}{2} - \pi^2 \right) + \frac{1}{\pi} \left(\frac{\pi^2}{2} + \pi^2 \right) = \pi$$

~~$\frac{1}{\pi} + \frac{1}{\pi} + 1 = \pi$~~

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_{-\pi}^{0} (-x-\pi) \cos nx\,dx + \frac{1}{\pi} \int_{0}^{\pi} (x+\pi) \cos nx\,dx$$

$$= \frac{1}{\pi} \left[(-x-\pi) \frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right) \right]_{-\pi}^{0} + \frac{1}{\pi} \left[(x+\pi) \frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right) \right]_{0}^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2} \right] + \frac{1}{\pi} \left[-\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{2}{n^2 \pi} \left[(-1)^n - 1 \right] = \frac{-4}{n^2 \pi} \quad \text{if } n \text{ is odd}$$

$= 0$ if n is even

3.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n + 1\right]$$

$$= \frac{2}{n}\left[1-(-1)^n\right] = \frac{4}{n} \text{ if } n \text{ is odd}$$

$$= 0 \text{ if } n \text{ is even}$$

Fourier series is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \cdots\right)$$

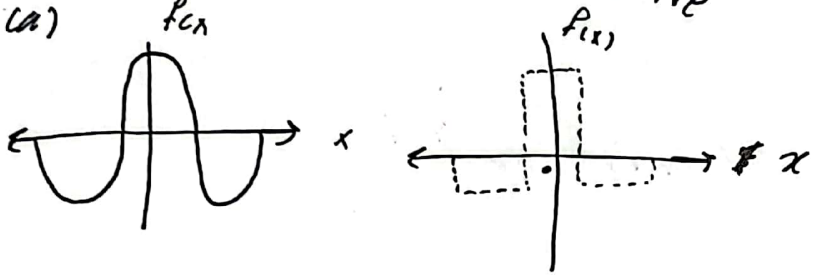
$$+ 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right)$$

Even function

A function $f(x)$ is said to be even (or symmetric) function if $f(-x) = f(x)$

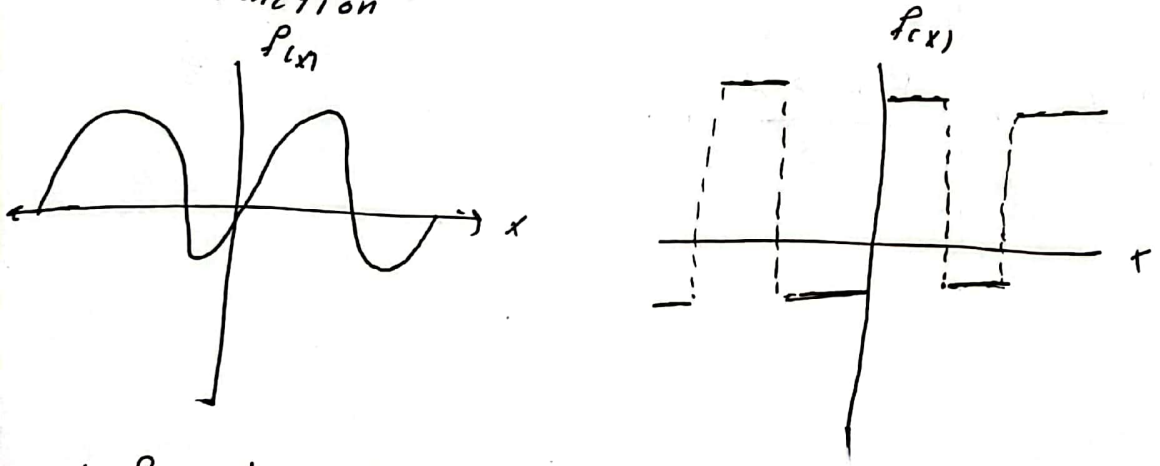
The graph of such a function is symmetric respect to y-axis [$f(x)$ axis]. Here y-axis is mirror or the reflection of the curve

(a)



The area under such a curve from $-\pi$ to π is double the area from of 0 to π

(b) odd function



A function $f(x)$ is called odd or (skew symmetric) function if $f(-x) = -f(x)$

Here the area under the curve from $-\pi$ to π is zero.

4/

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

- Expansion of an even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

As $f(x)$ and $\cos nx$ are both even function

∴ the product of $f(x)\cos nx$ is also even function

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

As $\sin nx$ is an odd function so $f(x)\sin nx$ is also an odd function the series of the even function will contain only cosine terms

- Expansion of an odd function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0 \quad [f(x) \cos nx \text{ is odd function}]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx$$

The series of the odd function will contain only sine terms

5/.

Ex: Find the Fourier series expansion of the periodic function of period of $f(x) = x^2 \quad -\pi < x < \pi$

Hence, Find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Sol:-

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + b_1 \sin x + \cdots$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} x^2\,dx = \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi} \left[\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) - (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi^2 \frac{\sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

Fourier series

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos x + \cdots$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right]$$

on putting $x = 0$ we have

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Ex :- Find the Fourier series for

$f(x) = x^3 \quad -\pi < x < \pi$

Sol :-

$f(x) = x^3$ is an odd function

$a_0 = a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \Rightarrow \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi} \left[x^3 \left(\frac{\cos nx}{n} \right) - 3x^2 \left(\frac{-\sin nx}{n^2} \right) + 6x \left(\frac{\cos n\pi}{n^3} \right) - 6 \left(\frac{\sin nx}{n^4} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3} \right)$$

$$= 2(-1)^n \left[\frac{-\pi^2}{n} + \frac{6}{n^3} \right]$$

$f(x) = b_1 \sin x + b_2 \sin 2x + \cdots$

$$x^3 = 2 \left(-\left(\frac{\pi^2}{1} + \frac{6}{1^3} \right) \sin x + \left(\frac{-\pi^2}{2} + \frac{6}{2^3} \right) \sin 2x \right.$$

$$\left. - \left(\frac{-\pi^2}{3} + \frac{6}{3^3} \sin 3x \right) + \cdots \right)$$

الموضوع : Appll التاريخ / / ٢٠٢

Solutions near an ordinary point

a) Solve the equation

$$y'' + 4y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^{n-2} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

replace n by n - 2

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 4 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\sum_{n=2} [n(n-1) a_n + 4 a_{n-2}] x^{n-2} \quad \dots 5$$

$$n(n-1) a_n + 4 a_{n-2} = 0 \quad \text{for } n \geq 2$$

$$a_n = \frac{-4 a_{n-2}}{n(n-1)} \quad \dots 6 \text{ recurrence relation}$$

$$a_2 = \frac{-4 a_0}{2 \cdot 1} \qquad a_3 = \frac{-4 a_1}{3 \cdot 2}$$

$$a_{2k} = \frac{-4 a_{2k-2}}{2k(2k-1)} \qquad a_{2k+1} = \frac{-4 a_{2k-1}}{(2k+1)(2k)}$$

التاريخ / / ٢٠٢

الموضوع :

$$a_2, a_4, \dots a_{2k} = \frac{(-1)^k 4^k}{(2k)!} a_0 \quad a_2 \dots a_{2k-2}$$

$$\text{So } a_{2k} = \frac{(-1)^k 4^k}{(2k)!} a_0 \quad k \geq 1 \quad \dots 7$$

$a_1 \neq 0$

نعوض بالسابق ونطلع

$$a_{2k+1} = \frac{(-1)^k 4^k a_1}{(2k+1)!} \quad \text{for } k \geq 1 \quad \dots 8$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 + \sum_{k=1}^{\infty} a_{2k} x^{2k} + a_1 x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k x^{2k}}{(2k!)} + a_1 \left[x + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k}{(2k+1)!} x^{2k+1} \right] \right]$$

التاريخ / / /

الموضوع :

$$(1-x^2)y'' - 6xy' - 4y = 0$$

$$\sum_{n=0} n(n+1)a_n x^{n-2} - \sum n(n-1)a_n x^n - 6\sum n a_n x^n - 4\sum a_n x^n = 0$$

$$\sum_{n=0} n(n-1)a_n x^{n-2} - \sum_{n=0} (n^2+5n+4)a_n x^n = 0$$

$$\sum_{n=0} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} (n+1)(n+4)a_n x^n = 0$$

replace n by n-2

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} (n-1)(n+2)a_{n-2} x^{n-2} = 0$$

$$a_n = \frac{n+2}{n} a_{n-2}$$

$$a_2 = \frac{4}{2} a_0 \qquad a_3 = \frac{5}{3} a_1$$

$$a_4 = \frac{6}{4} a_2 \qquad a_5 = \frac{7}{5} a_3$$

$$a_{2k} = \frac{2k+2}{2k} a_{2k+2} \qquad a_{2k+1} = \frac{2k+3}{2k+1} a_{2k-1}$$

$$a_2 a_4 a_6 \cdots a_{2k} = \frac{4 \cdot 6 \cdot 8 \cdots (2k+2)}{2 \cdot 4 \cdot 6 \cdots 2k} a_0 a_2 a_4 \cdots a_{2k-2}$$

التاريخ / / ٢٠٢

الموضوع :

$k \geq 1 \quad a_{2k} = (k+1)a_0$

$$a_{2k+1} = \frac{5 \cdot 7 \cdot 9 \cdots (2k+3)}{3 \cdot 5 \cdot 7 \cdots (2k+1)} a_1 a_3 \cdots a_{2k-1}$$

$$a_{2k+1} = \frac{2k+3}{3} a_1$$

$$y = a_0 + \sum_{k=1}^{\infty} a_{2k} x^{2k} + a_1 x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1}$$

$$y = a_0 + \sum_{k=1}^{\infty} (k+1) a_0 x^{2k} + a_1 x + \sum \frac{2k+3}{3} a_1 x^{2k+1}$$

$$y = a_0 \left[1 + \sum_{k=1}^{\infty} (k+1) x^{2k} \right] + a_1 \left[x + \sum \frac{2k+3}{3} x^{2k+1} \right]$$

4. Solve the equation

1) $y'' + 3xy' + 3y = 0$

$a_0 = 0 \longrightarrow a_0 \neq 0$

1/

Applied

Example: Evaluate the integral

$$I = \int_0^\infty x^4 e^{-x^3} dx$$

$u = x^3 \rightarrow u^{1/3} = x$, $du = 3x^2 dx \Rightarrow dx = \frac{1}{3} u^{-2/3} du$

$$= \frac{1}{3} \int_0^\infty u^{2/3} e^{-u} du = \frac{1}{3} \Gamma\left(\frac{2}{3}+1\right) = \frac{1}{3} \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{3} \cdot \frac{2}{3} \Gamma\left(\frac{2}{3}\right) = \frac{2}{9} \Gamma\left(\frac{2}{3}\right)$$

Ex2: $I = \int_0^1 x^2 \left(\ln \frac{1}{x}\right)^3 dx$

let $u = \ln x \Rightarrow e^u = x$

$x du = dx \Rightarrow e^u du = dx$

$$= \int_{-\infty}^0 e^{2u} (-u)^3 e^u du$$

$$= \int_0^\infty e^{3u} u^3 du$$

let $3u = v$ $\quad 3du = dv$

$du = \frac{1}{3} dv$

$$= \int_0^\infty e^{-v} \left(\frac{1}{3} v\right)^3 \frac{1}{3} dv$$

$$= \frac{1}{81} \int_0^\infty e^{-v} v^3 dv = \frac{1}{81} \Gamma(3+1) = \frac{1}{81} 3! = \frac{2}{27}$$

$$I = \int_0^1 \left(\ln \frac{1}{x}\right)^{p-1} dx$$

Let $u = \ln x \Rightarrow x = e^u$

$\therefore dx = e^u du$

Let $u = \ln \frac{1}{x} \Rightarrow \frac{1}{x} = e^u \Rightarrow x = \frac{1}{e^u} = e^{-u}$

$dx = -e^{-u} du$

$$= \int_{\infty}^{0} u^{p-1} \, -e^{-u} \, du$$

$$= \int_0^{\infty} e^{-u} u^{p-1} = \Gamma(p)$$

1/ Applied

Gamma function

using gamma function, evaluate the following integral $\sqrt{1/2}$

by definition $\int_0^\infty e^{-x} x^{n-1} dx$

$$\int_0^\infty e^{-t} \frac{1}{\sqrt{t}} dt$$

let $t = y^2$ then $dt = 2y\,dy$

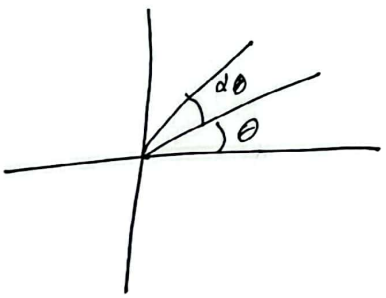
$$\int_0^\infty e^{-y^2} \frac{1}{y} 2y\,dy = 2\int_0^\infty e^{-y^2} dy$$

$$\text{let} = 2\int_0^\infty e^{-x^2} dx$$

$$\left(\sqrt{1/2}\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\,dx$$

~~let $r^2 = x^2 + y^2$~~ this integral can be easily evaluated in polar coordinates $r^2 = x^2 + y^2$, $A = dx\,dy$

$= r\,dr\,d\theta$



$$= 4 \int_0^{\infty} \int_0^{\pi/2} e^{-r^2} \, r \, dr \, d\theta$$

$$= 4 \int_0^{\infty} r e^{-r^2} dr \int_0^{\pi/2} d\theta$$

$$= 4 \int_0^{\infty} r e^{-r^2} dr \; [\theta]_0^{\pi/2}$$

$$= 4 \int_0^{\infty} r e^{-r^2} \frac{\pi}{2} \, dr$$

$$= -\frac{2}{2} \pi \int_0^{\infty} -2r e^{-r^2} dr = -\pi \left[e^{-r^2} \right]_0^{\infty} = \pi$$

$$\int_0^\infty e^{-h^2x^2}\,dx$$

let $u = h^2x^2$

$du = 2h^2x\,dx \Rightarrow dx = \frac{du}{2h^2x}$

$x = \frac{u^{1/2}}{h}$

$$\int_0^\infty e^{-u}\frac{1}{2h}u^{-1/2}\,du$$

$dx = \frac{1}{2h}u^{-1/2}\,dx$

$$= \frac{1}{2h}\int_0^\infty e^{-u}u^{-1/2}\,du = \frac{1}{2h}\Gamma\left(\frac{-1}{2}+1\right)$$

$$= \frac{1}{2h}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2h}\sqrt{\pi}$$

3) $\int_0^\infty x e^{-x^3} dx$

let $u = x^3 \rightarrow x = u^{\frac{1}{3}}$

$du = 3x^2 dx \Rightarrow dx = \frac{1}{3x^2} du$

$dx = \frac{1}{3} u^{-2/3} du$

$= \frac{1}{3} \int_0^\infty u^{1/3} u^{-2/3} e^{-u} du$

$= \frac{1}{3} \int_0^\infty u^{-1/3} e^{-u} du = \frac{1}{3} \Gamma(-\frac{1}{3}+1)$

$= \frac{1}{3} \Gamma(\frac{2}{3})$

prove that

$$\int_0^{\infty} e^{-x^4}\, dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

let $u = x^4 \Rightarrow u = x^{1/4}$

$$dx = \frac{1}{4} u^{-\frac{3}{4}}\, du$$

---

5) using gamma function evaluate the integral

$$\int_0^1 \sqrt[3]{\ln x}\, dx$$

---

remark

$$(-1)^{-1/2} = \left(e^{+i\pi}\right)^{-1/2}$$

$$= e^{-\frac{i\pi}{2}} = 1$$

Regular singular point:

Suppose that the point x_0 x_0 is a singular point of the equation $b_0(x)y'' + b_1(x)y' + b_2(x)y = 0$ --- (1) with polynomial coefficients, then $b_0(x_0) = 0$, So $b_0(x)$ has factor $(x - x_0)$ to some power.

let us put eq (1) into the form

$$y'' + p(x)y' + q(x)y = 0 \quad \text{---} \quad (2)$$

because $x = x_0$ is a singular point and because $p(x)$ & $q(x)$ are rational function of x, at least one (may be both) of $p(x)$ and $q(x)$ has denominator that contain the factor $(x - x_0)$. In what follows we assume that both $p(x)$ and $q(x)$ have been reduced so that in each case the numerator and denominator contain no common factors.

if $x=x_0$ is a singular point of eq (2), if the denominator (x) does not contain the factor $(x-x_0)$ to a power higher the one and if the denominator of $q(x)$ doesnot contain the factor $(x-x_0)$ to a power higher than two the $x=x_0$ is called Regular singular point (R.S.P) of eq (2)

but

if $x=x_0$ is a singular point ~~that~~ is not a Regular singular point, then it is called an irregular singular point (I.S.P)

الموضوع : التاريخ / / ٢٠٢٠

Ex: classify the singular points in the finite plane of the equation

1) $x(x-1)^2(x+2)y'' + x^2y' - (x^3+2x-1)y = 0$

$$y'' + \frac{x}{(x-1)^2(x+2)}y' - \frac{(x^3+2x-1)}{x(x-1)^2(x+2)}y = 0$$

$$y'' + p(x)y' - q(x)y = 0$$

$$p(x) = \frac{x}{(x-1)^2(x+2)} \qquad q(x) = \frac{(x^3+2x-1)}{x(x-1)^2(x+2)}$$

the singular points are $x = -2, 1, 0$

R.S.P $x = 0, -2$

I.S.P $x = 1$

2) $x^4(x^2+1)(x-1)^2 y'' + 4x^3(x-1)y' + (x+1)y = 0$

$$y'' + \frac{4}{x(x^2+1)(x-1)} y' + \frac{(x+1)}{x^4(x^2+1)(x-1)^2} y = 0$$

$y'' + P(x) y' + Q(x) y = 0$

the singular point in the finite plane

are $x = 0, 1, i, -i$

R.S.P at $x = 1, i, -i$

I.S.P at $x = 0$ because of the denominator $Q(x)$ higher than two

3) $y'' + xy = 0$

Since $y'' + P(x) y' + Q(x) y = 0$

$P(x) = 0$ and $Q(x) = x$

∴ No singular Points can be found in the finite plane

4) $(x^2+1)(x-4)^3 y'' + (x-4)^2 y' + y = 0$

$$y'' = \frac{1}{(x^2+1)(x-4)} y' + \frac{1}{(x^2+1)(x-4)^3} y = 0$$

the singular point in the finite plane are : $x = 4, i, -i$

R.S.P at $x = i, -i$

I.S.P at $x = 4$

because of the denominator of $q(x)$ is higher than two.

5) $x^3(x-1)y'' + (x-1)y' + 4xy = 0$

$$y'' + \frac{1}{x^3} y' + \frac{4}{x^2(x-1)} y = 0$$

the singular points in the finite plane are $x = 0, 1$

R.S.P at $x = 1$ I.S.P at $x = 0$

because of the denominator of $q(x)$ is higher than two.

$y^4 + \frac{1}{x^4} y = 0$ R.S.P = N.O I.S.P = O

1) $x^2(x^2-4)y'' + 2xy' + 3y = 0$

2) $(x^2-4)y'' + 2x^2y' + 3y = 0$

3) $x^2(x-2)y'' + 3y' + y = 0$

الموضوع :

التاريخ / / ٢٠٢٠

Fractional function

let us calculate the value of some integral for $\alpha > 0$

$$-\frac{1}{\alpha}\int_0^\infty -\alpha e^{-\alpha x} = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$$

next differentiate both sides of this eq repeatedly with respect to α

$$\int_0^\infty -x e^{-\alpha x}\, dx = -\frac{1}{\alpha^2}$$

$$\int_0^\infty x e^{-\alpha x}\, dx = \frac{1}{\alpha^2}$$

$$\int_0^\infty x^2 e^{-\alpha x}\, dx = \frac{2}{\alpha^3}$$

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

putting $\alpha = 1$ we get

$$\int_0^\infty x^n e^{-x} dx = n!$$

$$0! = \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = -e^{-\infty} + e^0 = 1$$

Gamma function

for n has been a non negative integer we will define the factorial function for non integer n by the definite integral (1)

The gamma function (Γ) and we will replace n by the letter p

So for any $p > 0$ $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx \quad p > 0$

for integer values of n

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = n-1!$$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!$$

$$\Gamma(1) = 0! = 1$$

$$\Gamma(2) = 1! = 1$$

$$\Gamma(3) = 2!$$

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx \quad p > 0 = p!$$

now let use integer the last equation by parts!

let $u = x^p$ $du = px^{p-1} dx$ $dv = e^{-x} dx$ $v = -e^{-x}$

$$uv - \int_0^\infty v\, du$$

$$-x^p e^{-x} \Big|_0^\infty - \int_0^\infty -e^{-x} p x^{p-1} dx$$

$$= p \int_0^\infty x^{p-1} e^{-x} dx = p\Gamma(p)$$

الموضوع :

التاريخ / / ٢٠٢٠

$\Gamma(p+1) = p\,\Gamma(p) \qquad p > 1$

$\Gamma(2.5) = 1.5\,\Gamma(1.5)$

$\Gamma(\frac{3}{2}) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\Gamma(\pi)$

$\Gamma(\frac{5}{2}) = \frac{5}{2}\Gamma(\frac{3}{2}+1) = \frac{5}{2}\frac{3}{2}\Gamma(\frac{1}{2}+1)$

$= \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})$

$\Gamma(p) = \frac{1}{p}\Gamma(p+1) \qquad p < 1$

$\Gamma(0.5) = \frac{1}{0.5}\Gamma(0.5+1) = \frac{1}{0.5}\Gamma(1.5)$

$\Gamma(-0.5) = \frac{1}{-0.5}\Gamma(-0.5+1) = \frac{1}{-0.5}\Gamma(0.5)$

Beta Function

The Beta Function is defined by a definite integral

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

Ex.1

$$B(6,5)=\int_0^1 t^5(1-t)^4\,dt$$

Ex.2

$$B(8,3)=\int_0^1 t^7(1-t)^2\,dt$$

Prove that $B(m,n)=B(n,m)$ one of the Properties Beta Function

L.H.S.

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

let $t=1-x \longrightarrow 1-t=x$

$$-dt=dx$$

$t: 0 \longrightarrow 1$

$x: 1 \longrightarrow 0$

$$= \int_1^0 (1-x)^{m-1} \; x^{n-1} \; -dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} \; dx$$

$$B(n,m) = R.H.S$$

To obtain the trigonometric form of Beta fun.

$$B(m,n) = \int_0^1 t^{m-1} (1-t)^{n-1} \; dt$$

let $t = \sin^2 x \implies 1-t = 1-\sin^2 x$

$$= \cos^2 x$$

$$dt = 2\sin x \cdot \cos x \, dx$$

$t : 0 \longrightarrow 1$

$x : 0 \longrightarrow \pi/2$

$$B(m,n) = \int_0^{\pi/2} (\sin^2 x)^{m-1} (\cos^2 x)^{n-1} \cdot 2\sin x \cos x \, dx$$

$$= \int_0^{\pi/2} (\sin x)^{2m-2} (\cos x)^{2n-1} \cdot 2\sin x \cos x \, dx$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} x \cdot \cos^{2n-1} x \, dx$$

$$B(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1} x \cdot \cos^{2n-1} x \, dx$$

ex.

$$B(5,4) = 2\int_0^{\pi/2} \sin^9 x \cos^7 x \, dx$$

$$= \int_0^1 t^4 (1-t)^3 \, dt$$

Another Form of Beta Function can be obtained by putting $x = \frac{y}{1+y}$

Then we get the Following Form of Beta Function

$$B(m,n) = \int_0^1 t^{m-1} (1-t)^{n-1} \, dt$$

let $t = \frac{y}{1+y} \longrightarrow 1-t = 1 - \frac{y}{1+y}$

$$1-t = \frac{1}{1+y}$$

$$dt = \frac{1}{(1+y)^2} dy$$

$t: 0 \longrightarrow 1$

$y: 0 \longrightarrow \infty$

$$B(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \cdot \left(\frac{1}{(1+y)^2}\right) dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n-2}} \cdot \frac{1}{(1+y)^2}\, dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

Ex

$$B(5,4) = \int_0^\infty \frac{y^4}{(1+y)^9}\, dy$$

Relation Between Gamma and Beta Functions

$$B(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$$

To prove the above relation, we have by defn. Beta and Gamma function:

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1} \, dx$$

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} \, dt$$

Put $t = x^2$, $dt = 2x\,dx$

$$\Gamma(n) = \int_0^{\infty} x^{2n-2} e^{-x^2} \cdot 2x \, dx$$

$$\Gamma(n) = 2\int_0^{\infty} x^{2n-1} e^{-x^2} \, dx$$

$$\Gamma(m) = 2\int_0^{\infty} y^{2m-1} e^{-y^2} \, dy$$

$$\Gamma(m+n) = 2\int_0^{\infty} r^{2(m+n)-1} e^{-r^2} \, dr$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^\infty\int_0^\infty x^{2n-1}\, y^{2m-1}\, e^{-(x^2+y^2)}\, dx\, dy$$

R.H.S. changing into polars

$x = r\cos\theta$, $y = r\sin\theta$

we have $x^2+y^2 = r^2$

$$dx\,dy = r\,dr\,d\theta$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1}\, r\,dr\,d\theta$$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2m+2n-1} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta\, dr$$

$$= \left[2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta\right]$$

$$= \left[2\int_0^\infty e^{-r^2} e^{2(m+n)-1}\, dr\right]\cdot\left[2\int_0^{\pi/2}\sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\, d\theta\right]$$

$\therefore \Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot B(m,n)$

$$\therefore B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Mathematical

(Chapter 2)

1. Factorial Function

Let us calculate the values of some integrals. For $\alpha > 0$,

$$\int_0^\infty e^{-\alpha x}\, dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty$$

$$= \frac{1}{\alpha}$$

Next differentiate both sides of this equation repeatdly with respect to α:

$$\int_0^\infty -x\, e^{-\alpha x}\, dx = -\frac{1}{\alpha^2}$$

$$\text{or} \int_0^\infty x\, e^{-\alpha x}\, dx = \frac{1}{\alpha^2}$$

$$\int_0^\infty x^2 e^{-\alpha x}\, dx = \frac{2}{\alpha^3}$$

$$\int_0^\infty x^3 e^{-\alpha x}\, dx = \frac{3!}{\alpha^4}$$

In general :

$$\int_0^\infty x^n \cdot e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

putting $\alpha = 1$ we get :

$$(1) \quad \ldots\ldots \int_0^\infty x^n \cdot e^{-x}\, dx = n! \qquad n = 1, 2, 3, \ldots$$

Thus we have a definite integral whose value is $n!$ for positive integer n.

We can use this integral to give a meaning to $0!$. putting $n = 0$ we get :

$$0! = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty$$

$$= -\left(e^{-\infty} - e^{0}\right) = 1$$

2. Gamma Function

So far n has been a nonnegative integer. We will define the factorial function for noninteger n by the definite integral (1).

It is customary to reserve the factorial notation for integer n and to call the corresponding function for noninteger n the gamma function (Γ) and we will

replace n by the letter p when we deal with nonintegers.

So, for any $p > 0$

$$\Gamma(p) = \int_0^\infty x^{p-1} \cdot e^{-x}\, dx \qquad p > 0$$

for integer values of n:

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$

$$= (n-1)!$$

and $$\Gamma(n+1) = \int_0^\infty x^n\, e^{-x}\, dx$$

$$= n!$$

$\therefore \quad \Gamma(1) = 0! = 1$

$\Gamma(2) = 1! = 1$

$\Gamma(3) = 2! = 2$

$\Gamma(4) = 3! = 6$

and so on.

Using p instead of n:

$$\Gamma(p+1) = \int_0^\infty x^p\, e^{-x}\, dx = p! \qquad p > -1$$

Now, Let us integrate the last equation by parts :

Let $x^p = u$ and $e^{-x}\,dx = dv$

$\therefore \quad du = p\,x^{p-1}\,dx$ and $v = -e^{-x}$

$$\therefore \quad \Gamma(p+1) = -x^p e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})\,p\,x^{p-1}\,dx$$

$$= p\int_0^\infty x^{p-1}\,e^{-x}\,dx$$

$$= p\,\Gamma(p)$$

The equation $\Gamma(p+1) = p\,\Gamma(p)$ is called recursion relation for the Γ function

Given Γ of any number $p > 0$, we can use it to find $\Gamma(p+1)$.

Γ function is usually given in tables for p between 1 and 2 and by using the recursion relation we can find $\Gamma(p)$ for p between 2 and 3.

For example :

$$\Gamma(2.5) = 1.5\,\Gamma(1.5)$$

Similarly, we have $\Gamma(3.5) = 2.5\,\Gamma(2.5)$ $= (2.5)(1.5)\,\Gamma(1.5)$ and so on.

To find $\Gamma(p)$ for p between 0 and 1 from the given values between 1 and 2 we write the recursion relation as:

$$\Gamma(p) = \frac{1}{p}\,\Gamma(p+1)$$

Then, for example,

$$\Gamma(0.5) = \frac{1}{0.5}\,\Gamma(1.5)$$

3. Gamma Function of negative numbers

For $p \leq 0$, $\Gamma(p)$ has not so far been defined.

We will use the recursion relation:

$$\Gamma(p) = \frac{1}{p}\,\Gamma(p+1)$$

For example

$$\Gamma(-0.5) = \frac{1}{-0.5}\,\Gamma(0.5)$$

$$\Gamma(-1.5) = \frac{1}{-1.5}\,\frac{1}{-0.5}\,\Gamma(0.5)$$

and so on.

Since $\Gamma(1) = 1$ We can see that:

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \to \infty \text{ as } p \to 0$$

4. Some important formulas involving Gamma function

First we evaluate $\Gamma\left(\frac{1}{2}\right)$

By definition:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\, dt$$

Note that it does not matter what letter we use for the dummy variable of integration in a definite integral

put $t = y^2$ then $dt = 2y\, dy$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{y} e^{-y^2} 2y\, dy$$

$$= 2\int_0^\infty e^{-y^2} dy$$

or $$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx$$

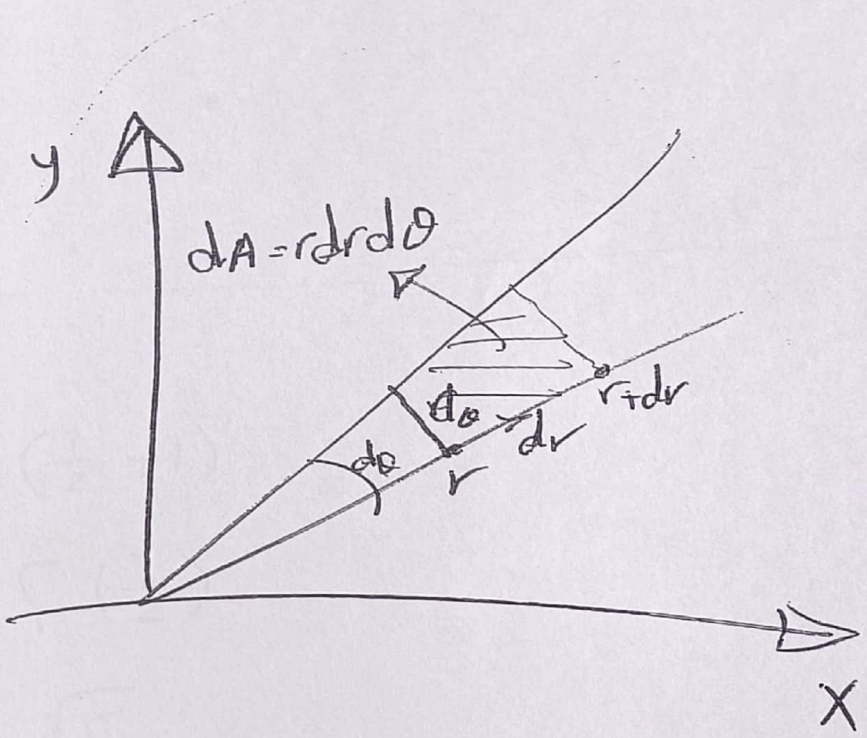
$$\therefore \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

This integral can be easily evaluated in polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$

$r^2 = x^2 + y^2$

$dx \cdot dy = dA$ area of small



$$\therefore \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4 \int_0^\infty r \cdot e^{-r^2}\, dr \int_0^{\pi/2} d\theta$$

$$= 4 \cdot \frac{\pi}{2} \left[\frac{e^{-r^2}}{-2}\right]_0^\infty = \pi$$

$\therefore \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

So we can prove such :

$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$

$\Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2}$$

2- Using Gamma function, evaluate $\int_0^\infty x^n e^{-h^2x^2}\,dx$

Let $h^2x^2 = u$, $\quad du = 2xh^2\,dx$

$x^n = u^{\frac{n}{2}}$, $\quad 2x = \frac{1}{h^2}\,du$

The integral becomes $\int_0^\infty e^{-u} \cdot \frac{1}{h^2}\left(\frac{u}{h^2}\right)^{\frac{n}{2}} du$

$= \frac{1}{2h^{n+1}} \int_0^\infty u^{\frac{n}{2}} e^{-u}\,du$

$= \frac{1}{2h^{n+1}} \Gamma\left(\frac{n}{2}+1\right) = \frac{1}{2h^{n+1}} \Gamma\left(\frac{n}{2}\right)$

3- Use Gamma Function to evaluate the integral:

$$\int_0^\infty x\, e^{-x^3}\, dx$$

Let $x^3 = u$ $\quad 3x^2 dx = du$

$$\therefore dx = \frac{1}{3} x^{-2}\, du = \frac{1}{3} \cdot u^{-\frac{2}{3}}\, du$$

∴ The integral becomes:

$$= \frac{1}{3} \int_0^\infty u^{\frac{1}{3}} \cdot u^{-\frac{2}{3}} \cdot e^{-u}\, du$$

$$= \frac{1}{3} \int_0^\infty u^{-\frac{1}{3}}\, e^{-u}\, du$$

$$= \frac{1}{3} \cdot \Gamma\left(-\frac{1}{3}+1\right) = \frac{1}{3} \cdot \Gamma\left(\frac{2}{3}\right)$$

$$= \frac{1}{3} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{2}{3}+1\right)$$

$$\approx \frac{1}{2} \cdot \Gamma(1.66) \approx \frac{1}{2} * 0.90167$$

4- prove that $\int_0^\infty e^{-x^4} dx = \frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right)$

Let $x^4 = u$ $\quad 4x^3 dx = du$

$$\therefore dx = \frac{1}{4} x^{-3}\, du = \frac{1}{4} \cdot u^{-\frac{3}{4}}\, du$$

The integral becomes: $\frac{1}{4} \int_0^\infty e^{-u}\, u^{-3/4}\, du$

$$= \frac{1}{4}\Gamma\left(-\frac{3}{4}+1\right)$$

$$= \frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right).$$

5- Using Gamma Function, evaluate the integral:

$$\int_0^1 \sqrt[3]{\ln x}\cdot dx$$

Let $\ln x = u \qquad x = e^u$

$\therefore dx = e^u\, du$

When $x = 0 \qquad u = +\infty$

$x = 1 \qquad u = 0$

$\therefore$ The integral becomes:

$$\int_{-\infty}^{0} u^{\frac{1}{3}}\cdot e^u\, du = -\int_0^{\infty} u^{\frac{1}{3}}\cdot e^u\cdot du.$$

Let $u = -t \qquad du = -dt$

$\therefore$ The integral becomes: $-\int_0^{\infty} (-t)^{\frac{1}{3}}\cdot e^{-t}(-dt)$

$$= -\int_0^{\infty} t^{\frac{1}{3}} e^{-t}\, dt = -\Gamma\left(\frac{1}{3}+1\right)$$

$$= -\frac{1}{3}\cdot\Gamma\left(\frac{1}{3}\right).$$

note: $(-1)^{-1/2} = (e^{i\pi})^{-1/2}$

$= e^{-i\pi/2}$

$= 1$

Home Work:

Use Gamma Function to evaluate the following integrals:

1- $\int_0^{\infty} x^4 \cdot e^{-x^3} \cdot dx$

2- $\int_0^{1} x^2 \cdot \left(\ln \frac{1}{x}\right)^3 \cdot dx$

3- $\int_0^{1} \left(\ln \frac{1}{x}\right)^{p-1} \cdot dx$

4- $\int_0^{\infty} x^{-\frac{1}{3}} \cdot e^{-8x} \cdot dx$

5- $\int_0^{\infty} x^6 \cdot e^{-2x} \cdot dx$

6- $\int_0^{\infty} x^3 \cdot e^{-x} \cdot dx$

7- $\int_0^{\infty} x^2 \cdot e^{-x^2} \cdot dx$

8- $\int_0^{\infty} y^{2z-1} \cdot e^{-y^2} \cdot dy$

تالت رياضيات ابلايت
حسن ورود رياضي

Power Series Solution.

Consider the homogeneous linear equation of the second order,

$$b_0(x)y'' + b_1(x)y' + b_2(x)y = 0 \quad \text{——— (1)}$$

with polynomial coefficients if $b_0(x)$ does vanish at $x=0$, then in some interval about x (تنتهي), staying aways from the nearst point where (الابتعاد عن اقرب نقطة) $b_0(x)$ does vanish, its safe to divide through by $b_0(x)$, thus we replace eq. (1) by:

$$y'' + \frac{b_1(x)}{b_0(x)}y' + \frac{b_2(x)}{b_0(x)}y = 0 \quad \text{——— (*)}$$

implice,

$$y'' + p(x)y' + q(x)y = 0$$

دوال كسرية

in which the coefficient $p(x), q(x)$ are Rational functions of x with denomiators that do not vanish at $x=0$.

مقام ≠ صفر

Real number: ١, ٤, ٥, -١, -٢, $\sqrt{٤}$, π, $\frac{٣}{٥}$

حقل الاعداد الحقيقية

Convergence of Power Series :-

From Calculas we know that the Power Series $\sum_{n=0}^{\infty} a_n x^n$ Converges either at $x=0$ only or for all finite x or the series converges in an interval $-R < x < R$ and diverges outside the interval. Unless the series converges at only one point, it represent, where it does converge, a function $f(x)$ in the sense that the series has at each value x the sum $f(x)$.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then also

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad -R < x < R$$

$$\int_0^x f(y)\,dy = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}; \quad -R < x < R$$

} all of the

that is, the series is termwise, differential and integrable in the sense that ⊛, the series of derivative of the separate terms converges to derivative of the sum of the original series

Ex(1): From Calculus:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad -1 < x < 1 \quad \text{——①}$$

Misbehavior of the function $f(x) = \frac{1}{1-x}$ at $x = 1$

the interval of convergence terminates at $x = 1$.

Now, replace x by $(-x^2)$ to get:

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \Rightarrow$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad , \quad -1 < x < 1 \quad \text{——②}$$

again the interval of convergence is $-1 < x < 1$,

but the function $f(x) = \frac{1}{1+x^2}$ well behaved for

all real x.

let x be a complex variable to understand what

going on here?

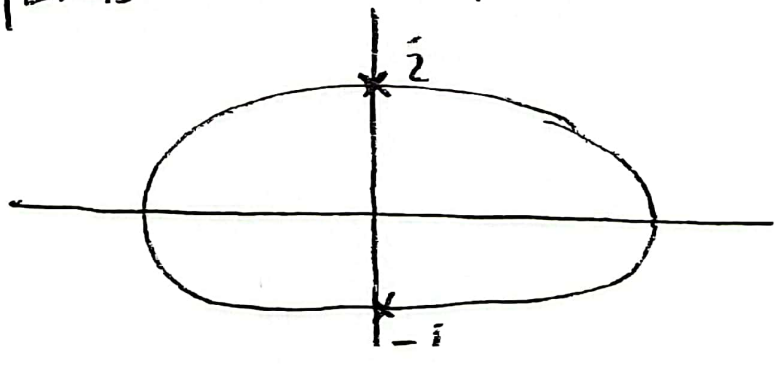
The points $x = i$ and $x = -i$, where the denom

$(1+x^2)$ vanis. Thus the interval of convergese f

eq. (2) is all values of x inside the circle sh

in Fig (1) {Exist in books of function of Complex

variables}.



The Power Series in a complex variable x always has its region of convergence the interior of a circ The circle of convergence of Power Series cannot have inside it a singularity of the function represe by the series. The circle of convergence has its cen at the origin and passes ~~through~~ through the Singularity nearest the origin.

Ex(2)% let, $f(x) = \dfrac{x-2}{(x-3)(x-4)(x^2+25)}$

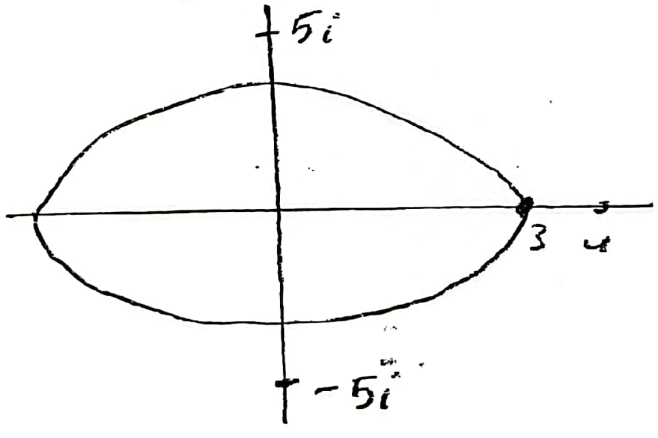
Solution%

[661]

The denominator which vanishes at

$$x = 3, 4, 5i, -5i$$

The function $f(x)$ has a power series expansion valid inside the circle Fig (2) with center at x= and extending as far as the nearest point $x=3$ where the function misbehaves. For all real x the interval of convergence in $-3 < x < 3$.



نأخذ معادلة تفاضلية خطية من الدرجة الثانية

$$b_0(x)y'' + b_1(x)y' + b_2(x)y = 0 \quad \text{---} (1)$$

نقسم على $b_0(x)$ بشرط أن $b_0(x) \neq 0$ فنحصل على

$$y'' + P(x)y' + q(x)y = 0 \quad \text{---} (2)$$

إذا كان $P(x)$, $q(x)$ تحليلية (analytic) عند $x = x_0$ ← ordinary

إذا لم تكن تحليلية ← singular

مثال ordinary Point

$$y' + y = 0$$

مثال singular Point

$$x^2y'' + y = 0$$

إذن

$$q(x) = \frac{1}{x^2}$$

ordinary Points and Singular Points

For a linear differential equation:

$$b_0(x)y^{n} + b_1(x)y^{n-1} + \cdots + b_n(x)y = R(x) \quad \text{---}$$

with Polynomials Coefficients, the Point $x = x_0$ is Called an ordinary Point of the equation if $b_0(x_0) \neq 0$.

A Singular Point of the Linear eq. (1) is any Point $x = x_1$ for Which $b_0(x_1) = 0$.

In this Section we Shall obtain Power series Solution Valid near an ordinary Point of a linear e and in next Section we Shall get Power Series Solution Valid near a certain Kind of Singular Point of the eq. (1).

Defn.

ordinary and Singular Points

Consider the linear differential eq. in the Standard For $y'' + p(x)y' + q(x)y = 0$. A Poin x_0 is Called an ordinary Point if both p, q are analytic at x_0. If x_0 is not an ordinary Point, It is called Singular Point of the eq.

Ex(3): Find the Singular points for each eq.'s in the finite plane.

Solution

1- $(1-x^2)y'' - 6xy' - 4y = 0$

has $x=1$ and $x=-1$
as its only Singular Points in the finite compl

2- $y'' + 2xy' + y = 0$

has no Singular Points in the finite plane.

3- $xy'' + y' + xy = 0$

has the origin $x=0$, as only Singular Point in the finite plane.

Exercises To Solve [1780

For each equation, list all of the Singular Points in the finite plane.

1- $(x^2+4)y'' - 6xy' + 3y = 0$

2- $x(3-x)y'' - (3-x)y' + 4xy = 0$

3- $4y'' + 3xy' + 2y = 0$

4- $x(x-1)^2 y'' + 3xy' + (x-1)y = 0$

5- $x^4 y'' + y = 0$

6- $(x^2-4x+3)y'' + x^2 y' - 4y = 0$

7- $(1+x^2)y'' - 2xy' + 6y = 0$

8- $(4x+1)y'' + 3xy' + y = 0$

9- $x^2(1+4x^2)y'' - 4xy' + y = 0$

10- $x^3(x^2-4)^2 y'' + 2(x^2-4)y' - xy = 0$

11- $(x^2+6x+8)y'' + 3y = 0$

12- $4xy'' + y = 0$

13- $x^2(1-x)^3 y'' + (1+2x)y = 0$